On The Khintchine Constant For Centred Continued Fraction Expansions∗

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Abstract

In this note, we consider a classical constant that arises in number theory, namely the Khintchine constant. This constant is closely related to the growth of partial quotients that appear in continued fraction expansions of reals. It equals the limit of the geometric mean of the partial quotient which is proved to be the same for almost all real numbers. We provide several expressions for this constant in the particular case of centred continued fraction expansions as well as a numerical evaluation of this constant up to 1000 digits.

1 Introduction

All real numbers admit various expansions into continued fractions. Here, two different continued fraction expansions are presented, namely, the standard continued fraction expansion and the centred continued fraction expansion (see Rockett and Szüsz [12] for a precise presentation of standard continued fraction expansions and Schweiger [13] for a description of a large class of continued fraction expansions).

The growth of the partial quotients (i.e., the continued fraction “digits”) that appear in the expansion is a particularly interesting subject. Khintchine [6] has proved the strong fact that for almost all real numbers, the geometric mean of these partial quotients tends to a constant, namely the Khintchine constant.

This constant has been extensively studied in the case of standard continued fraction expansions. The current record for its numerical evaluation is owned by Gourdon [5] who gives its first 110,000 digits.

This note answers a question of Finch regarding the centred continued fraction constant. Several alternative expressions for this constant are provided. This enables an evaluation of its first 1000 digits. Independently, Adamshik has evaluated the first 250 digits of the centred Khintchine constant.

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2 Continued Fraction Expansions

In this section, we recall the classical definition and the main properties of the continued fraction expansion based on the classical Euclidean algorithm, namely the standard continued fraction expansion. The definition given here is by means of expanding maps. Then, we recall some useful properties of the expansion in order to compute the Khintchine constant.

Next, we present a slightly different continued fraction expansion based on the Euclidean algorithm to the nearest integer, namely the centred continued fraction expansion. The maps used in its definition are just a kind of translation of the standard maps. Furthermore, this expansion satisfies some similar properties that enable us to compute the centred Khintchine constant.

2.1 Standard continued fraction expansions

First, consider the standard continued fraction expansion of a real number $0 < x \leq 1$,

$$x = \frac{1}{q_1 + \frac{1}{q_2 + \frac{1}{q_3 + \cdots}}} = [q_1, q_2, q_3, \ldots],$$

where $q_1, q_2, q_3, \ldots$ are strictly positive integers.

The sequence $(q_1, q_2, q_3, \ldots)$ of partial quotients in the expansion can be obtained by the shift function $T : [0, 1] \to [0, 1]$ and the map function $\sigma : [0, 1] \to \mathbb{N}$,

$$T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \quad \sigma(x) = \left\lfloor \frac{1}{x} \right\rfloor.$$

The sequence $M(x) := (q_1, q_2, q_3, \ldots)$ of partial quotients that intervene in the expansion of the real number $x$ is equal to $(\sigma(x), \sigma(T(x)), \sigma(T^2(x)), \ldots)$.

As proved by Kuzmin in [7], the probability $m_n(t)$ that the expansion $M(x) := (q_1, q_2, q_3, \ldots)$ of a number $x \in [0, 1]$ satisfies $q_n \geq 1/t$ converges to the function $m(t) := \log(1 + t)/\log(2)$. This property is usually referred to as the Gauss-Kuzmin theorem. At the same time, Lévy in [9] proved the same theorem using a completely different method. The probability density $p(t) := 1/(\log(2)(1 + t))$ whose distribution function is $m(t)$ is usually known as Gauss’ measure.

This theorem gives access to the frequency $f_m$ of the digit $m$ upon integrating the Gauss’ measure $p(x)$ over the interval $[1/(m + 1); 1/m]$,

$$f_m = \frac{1}{\log 2} \log(1 + \frac{1}{m(m + 2)}).$$

Finally, Khintchine in [6] proved that for almost all real $x$, the geometric mean of the quotients in the continued fraction expansion of $x$ tends to a constant whose expression is

$$K_{SCF} := \lim_{n \to \infty} \sqrt[n]{q_1 q_2 \ldots q_n} = \prod_{m=1}^{\infty} \left(1 + \frac{1}{m(m + 2)}\right)^{\frac{\log m}{m+2}} \approx 2.685452001 \ldots.$$
This infinite product converges very slowly. Lehmer [8], Shanks and Wrench [14, 15], Gosper, Bailey, Borwein, and Crandall [1] provide several representations that make possible a precise numerical evaluation of $K_{SCF}$. The record currently belongs to Gourdon [5] who has determined the constant to 110,000 decimal places.

2.2 Centred continued fraction expansions

The principle of the centred Euclidean algorithm is to consider a pseudo-Euclidean division that involves the nearest integer rounding function $\lceil x \rceil := \lfloor x + \frac{1}{2} \rfloor$.

This corresponds to a continued fraction expansion for a real $-1/2 < x \leq 1/2$ of the form

$$x = \frac{\varepsilon_1}{q_1 + \frac{\varepsilon_2}{q_2 + \frac{\varepsilon_3}{q_3 + \ldots}}} = \left[ \varepsilon_1 q_1, \varepsilon_2 q_2, \varepsilon_3 q_3, \ldots \right],$$

(1)

where $\varepsilon_i = \pm 1$, and $q_1, q_2, q_3, \ldots$ are strictly positive integers.

Precisely, the sequence $(\varepsilon_1 q_1, \varepsilon_2 q_2, \varepsilon_3 q_3, \ldots)$ of the partial quotients in the expansion is obtained by a combination of iterations of the shift function $T : \left( \left[-1/2, 1/2\right] \setminus \{0\} \right) \rightarrow \left( \left[-1/2, 1/2\right] \setminus \{0\} \right)$ and the map function $\sigma : \left( \left[-1/2, 1/2\right] \setminus \{0\} \right) \rightarrow \mathbb{N}$, defined as follows

$$T(x) = \left\lfloor \frac{1}{x} \right\rfloor - \left\lfloor \left\lfloor \frac{1}{x} \right\rfloor \right\rfloor, \quad \text{and} \quad \sigma(x) = \left\lfloor \frac{1}{x} \right\rfloor.$$

This, together with the sign function $\text{sgn}(x)$ provide the sequences $(q_1, q_2, q_3, \ldots)$ and $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots)$ associated to the real $x$ in (1)

$$(q_1, q_2, q_3, \ldots) = (\sigma(x), \sigma(T(x)), \sigma(T^2(x)), \ldots), \quad \text{and} \quad$$

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots) = (\text{sgn}(x), \text{sgn}(T(x)), \text{sgn}(T^2(x)), \ldots).$$

In [10], Rieger proves a Gauss-Kuzmin theorem for the centred continued fraction expansion. The expansion (1) admits an invariant density of the Gauss’ measure type. This measure can be found in Rieger [10].

PROPERTY 1. (Rieger) The invariant measure of the centred continued fraction expansion has density

$$p(x) = \begin{cases} 
\frac{1}{\log \phi} + \frac{1}{\phi^2 + x} & \text{if } -1/2 \leq x < 0, \\
\frac{1}{\log \phi} - \frac{1}{\phi + x} & \text{if } 0 \leq x \leq 1/2, 
\end{cases}$$

where $\phi := \frac{1 + \sqrt{5}}{2}$.

By integrating this density for both the positive and the negative case, one obtains the frequency $f_m$ of the digit $m$. 
PROPERTY 2. (Rockett) The frequency $f_m$ of digit $m$ equals

$$ f_m = \begin{cases} \frac{1}{\log \phi} \log \frac{3 + 5\phi}{2 + 5\phi}, & \text{if } m = 2, \\ \frac{1}{\log \phi} \log \left( \frac{\phi(m - \frac{1}{2}) + 1 \phi^2(m + \frac{1}{2}) - 1}{\phi(m + \frac{1}{2}) + 1 \phi^2(m - \frac{1}{2}) - 1} \right), & \text{otherwise.} \end{cases} $$

This expression for the frequency of digit $m$ provides a representation of the Khintchine constant for the centered continued fraction expansion. This Khintchine constant is defined as the almost sure limit of the geometric mean of the absolute values of partial quotients in the centered continued fraction expansion.

COROLLARY. (Rockett) The centered Khintchine constant admits the following expression

$$ K_{CCF} = \frac{3 + 5\phi}{2 + 5\phi} \prod_{m \geq 3} \left( \frac{\phi(m - \frac{1}{2}) + 1 \phi^2(m + \frac{1}{2}) - 1}{\phi(m + \frac{1}{2}) + 1 \phi^2(m - \frac{1}{2}) - 1} \right)^{\frac{\log m}{\log \phi}}. $$

This expression as well as the expression for the frequency of the digits have been given by Rockett in [11].

The infinite product converges very slowly. We give in the sequel several alternative expressions of this constant in order to obtain a precise numerical evaluation of $K_{CCF}$.

3 Evaluation of $K_{CCF}$

3.1 Expression of $K_{CCF}$ involving the $\zeta'$ function

First, remark that the frequency $f_m$ is the value at $1/m$ of a complex function $\psi(z) := \sum_{n \geq 2} a_n z^n$, that is analytic at 0. This leads to an expression of $K_{CCF}$ by means of the derivative of the Riemann zeta function,

$$ \log \phi \log K_{CCF} = \log 2 \log \left( \frac{3 + 5\phi}{2 + 5\phi} \right) - \log \phi \sum_{n \geq 2} a_n (\zeta'(n) - \frac{\log 2}{2^n}), $$

where $\zeta'(n) := -\sum_{m \geq 1} \log(m)/m^n$. It proves convenient to introduce an integer parameter $N$ in order to decrease the number of $\zeta'$ evaluations.

PROPOSITION 1. Let $N$ be an arbitrary positive integer. The centered Khintchine constant is expressible in terms of the $\zeta'$ tail function

$$ \zeta'(n, N) := \sum_{i = N+1}^{\infty} \frac{\log i}{i^n} = \zeta'(n) - \sum_{i = 2}^{N} \frac{\log i}{i^n} $$

under the form

$$ \log \phi \log K_{CCF} = \log 2 \log \left( \frac{3 + 5\phi}{2 + 5\phi} \right) + \sum_{m = 3}^{N} f_m \log(m) - \sum_{n \geq 2} a_n \zeta'(n, N), $$

(3)
where \(\psi(1/m) := \sum_{n \geq 2} a_n / m^n\) is the expansion of \(f_m\).

A fast approximation of the first 10 digits of \(K_{CCF}\) is obtained by taking \(N = 2000\), \(a_2 = 2.078086920\) and \(a_3 = -0.490569760\). Due to lack of sufficiently fast algorithms dedicated to computing the values of the \(\zeta'\) function, it proves useful to deal with the Riemann zeta function instead of its derivative.

### 3.2 Expression of \(K_{CCF}\) involving the \(\zeta\) function

We give here an expression of \(K_{CCF}\) by means of the Riemann \(\zeta\) function for which fast evaluation algorithms are known (see Borwein [2]).

**THEOREM 1.** The centred Khintchine constant is expressible in terms of the \(\zeta\) function

\[\zeta(n) := \sum_{i=1}^{\infty} \frac{1}{i^n}\]

under the form

\[
\log \phi \log K_{CCF} = \log 3 \log \phi + \log \frac{2}{3} \log \left(\frac{5\phi + 3}{2\phi + 2}\right)
\]

\[+ \sum_{n=2}^{\infty} \frac{(-1)^n}{n} (\zeta(n) - 1) - \frac{1}{2^n} \left[ \lambda_1^n h_n(\frac{1}{\lambda_1}) + h_n(\lambda_1) - \lambda_2^n h_n(\frac{1}{\lambda_2}) - h_n(\lambda_2) \right]
\]

where \(\lambda_1 := (\phi + 2)/(2\phi)\) and \(\lambda_2 := 1/(2\phi^3)\) involve the golden ratio \(\phi := (\sqrt{5} + 1)/2\) and \(h_n\) is the harmonic function

\[h_n(x) := \sum_{k=1}^{n-1} \frac{x^k}{k}.\]

**PROOF.** First, Abel’s summation formula

\[A_N = S_{N+1} b_N - \sum_{k=3}^{N} S_n (b_{n+1} - b_n),\]

with

\[S_n := \sum_{k=3}^{n} \log \left(\frac{(k - \frac{1}{2}) + \frac{1}{\phi} (k + \frac{1}{2}) - \frac{1}{2\phi}}{(k + \frac{1}{2}) + \frac{1}{\phi} (k - \frac{1}{2}) - \frac{1}{2\phi}}\right), \quad b_n := \log n,
\]

applies to the partial sum \(A_N\) of the second term in the expression (2) of \(K_{CCF}\). The sum \(S_n\) simplifies to

\[S_n = \log \left(\frac{\phi}{2\phi + 2} \frac{(n + \frac{3}{2}) - \frac{1}{\phi}}{(n + \frac{3}{2}) + \frac{1}{\phi}}\right) = \log \left(\frac{\phi}{2\phi + 2} \left(1 + \frac{1}{n+1}\right)\right).
\]
Thus,

\[ A_N = \log \left( \frac{5\phi + 3}{2\phi + 2} \right) \left( \log N - \sum_{k=3}^{N} \log(1 + \frac{1}{k}) \right) + \log \left( \frac{(N + \frac{3}{2}) - \frac{1}{\phi^3}}{(N + \frac{3}{2}) + \frac{1}{\phi}} \right) \log N \]

\[ + \sum_{k=3}^{N} \log(1 + \frac{1}{k}) \left[ \log(1 + (\frac{1}{2\phi^3}) \frac{1}{k}) - \log(1 + (\frac{\phi + 2}{2\phi}) \frac{1}{k}) \right]. \]

(5)

Then, by taking the limit when \( N \) tends to \( \infty \), one has

\[ \lim_{N \to \infty} (\log N - \sum_{k=3}^{N} \log(1 + \frac{1}{k})) = \log 3, \quad \text{and} \quad \lim_{N \to \infty} \log N \log \left( \frac{(N + \frac{3}{2}) - \frac{1}{\phi^3}}{(N + \frac{3}{2}) + \frac{1}{\phi}} \right) = 0. \]

Finally, the last term of the summation (5) involves two terms of the form \( \log(1 + x) \log(1 + \lambda x) \) with \( \lambda_1 = (\phi + 2)/(2\phi) \) and \( \lambda_2 = 1/(2\phi^3) \). This term admits the expansion

\[ \log(1 + x) \log(1 + \lambda x) = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \left[ \lambda^n h_n(\frac{1}{\lambda}) + h_n(\lambda) \right] x^n, \quad \text{where} \quad h_n(x) := \sum_{k=1}^{n-1} \frac{x^k}{k}. \]

This leads to formula (4) in the statement of Theorem 1.

Notice that the Leibniz theorem for alternating series applies. Thus, an approximation of the Khintchine constant, upon using the first \( n \) terms of the sum implies an error term of the form \( \rho^n \) with \( \rho < 0.56 \). Thus each new term adds about \( 1/3 \) of a digit.

An integer parameter \( N \) can be introduced in order to decrease the number of evaluations of the zeta function as was indicated earlier for the formula (3) in the context of the \( \zeta' \) function. We have

\[ \log \phi \log K_{CCF} = \log 3 \log \phi + \log \frac{2}{3} \log \left( \frac{5\phi + 3}{2\phi + 2} \right) \]

\[ + \sum_{k=3}^{N} \log(1 + \frac{1}{k}) \left[ \log(1 + \frac{\lambda_1}{k}) - \log(1 + \frac{\lambda_2}{k}) \right] \]

\[ + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n, N + 1) \left[ \lambda_1^n h_n(\frac{1}{\lambda_1}) + h_n(\lambda_1) - \lambda_2^n h_n(\frac{1}{\lambda_2}) - h_n(\lambda_2) \right], \]

(6)

where

\[ \lambda_1 := \frac{\phi + 2}{2\phi}, \quad \lambda_2 := \frac{1}{2\phi^3}, \quad \phi := \frac{\sqrt{5} + 1}{2}, \]

\[ h_n(x) := \sum_{k=1}^{n-1} \frac{x^k}{k}. \]
There $\zeta(n, N)$ is the standard Hurwitz function

$$\zeta(n, N) = \sum_{i=N}^{\infty} \frac{1}{i^n}.$$  

This trick has been previously used by Flajolet and Vardi [4] in the context of the standard Khintchine constant.

### 3.3 Numerical Evaluation

The expression (6) of $K_{CCF}$ allows a fast computation of the centred Khintchine constant to 1000 digits. Take $N = 20$ and 900 terms of the $m$ in (6) and get:

5.454517244545585756966057724994381016973272416251347045398035204159
848149224534457046551892428236520890860464032378984988603157831225610
06465979154678924336256871870147200595918162772167556536721579206031
81375840007159401994734031863260737005788373341011046964689121709296
108085664253384918627002328768243615809078241454228858477373388452
637551074162384500833786545687821051091444913535550458758504694557615
1526024529907215944083910506539103203453734297572686592339909645879
467559587716999010968167906220522783671194035940320571956005074825
3459834247918399855450907761112812630604425852979159496610236385270
0989385673791927720475422791641994398337283475772784382908662631354
227597610906502052382038440930720267454249413386781230763447006866
643010618553705813074959769600637242752799178902053811502778601186
14316797042073503878690050633187009534069269541813275117635845989159
97305420785624441235023652394439866214446551911196520147097949453518
8403499143182608393739420553268047580172019979620

The computation needs about $1 \cdot 10^{11}$ elementary operations (3 minutes on a 500 Mhz machine in 2001).

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### References


http://pauillac.inria.fr/algo/bsolve/constant/constant.html.


