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On The Khintchine Constant For Centred Continued Fraction Expansions*

Jérémie Bourdon†

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Abstract

In this note, we consider a classical constant that arises in number theory, namely the Khintchine constant. This constant is closely related to the growth of partial quotients that appear in continued fraction expansions of reals. It equals the limit of the geometric mean of the partial quotient which is proved to be the same for almost all real numbers. We provide several expressions for this constant in the particular case of centred continued fraction expansions as well as a numerical evaluation of this constant up to 1000 digits.

1 Introduction

All real numbers admit various expansions into continued fractions. Here, Two different continued fraction expansions are presented, namely, the *standard continued fraction expansion* and the *centred continued fraction expansion* (see Rockett and Szüsz [12] for a precise presentation of standard continued fraction expansions and Schweiger [13] for a description of a large class of continued fraction expansions).

The growth of the partial quotients (i.e., the continued fraction "digits") that appear in the expansion is a particularly interesting subject. Khintchine [6] has proved the strong fact that for almost all real numbers, the geometric mean of these partial quotients tends to a constant, namely the Khintchine constant.

This constant has been extensively studied in the case of standard continued fraction expansions. The current record for its numerical evaluation is owned by Gourdon [5] who gives its first 110,000 digits.

This note answers a question of Finch regarding the centred continued fraction constant. Several alternative expressions for this constant are provided. This enables an evaluation of its first 1000 digits. Independently, Adamshik has evaluated the first 250 digits of the centred Khintchine constant.

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†LINA, CNRS FRE 2729 Université de Nantes, France

2 Continued Fraction Expansions

In this section, we recall the classical definition and the main properties of the continued fraction expansion based on the classical Euclidean algorithm, namely the *standard continued fraction expansion*. The definition given here is by means of expanding maps. Then, we recall some useful properties of the expansion in order to compute the Khintchine constant.

Next, we present a slightly different continued fraction expansion based on the Euclidean algorithm to the nearest integer, namely the *centred continued fraction expansion*. The maps used in its definition are just a kind of translation of the standard maps. Furthermore, this expansion satisfies some similar properties that enable us to compute the centred Khintchine constant.

2.1 Standard continued fraction expansions

First, consider the standard continued fraction expansion of a real number $0 < x \leq 1$,

$$x = \cfrac{1}{q_1 + \frac{1}{q_2 + \frac{1}{q_3 + \cdots}}} = [q_1, q_2, q_3, \ldots],$$

where $q_1, q_2, q_3, \ldots$ are strictly positive integers.

The sequence $(q_1, q_2, q_3, \ldots)$ of partial quotients in the expansion can be obtained by the shift function $T :]0,1] \to]0,1]$ and the map function $\sigma :]0,1] \to \mathbb{N}$,

$$T(x) = \frac{1}{x} - \left[\frac{1}{x}\right], \qquad \sigma(x) = \left[\frac{1}{x}\right].$$

The sequence $M(x) := (q_1, q_2, q_3, \ldots)$ of partial quotients that intervene in the expansion of the real number x is equal to $(\sigma(x), \sigma(T(x)), \sigma(T^2(x)), \ldots)$.

As proved by Kuzmin in [7], the probability $m_n(t)$ that the expansion $M(x) := (q_1, q_2, q_3, \ldots)$ of a number $x \in [0, 1)$ satisfies $q_n \geq 1/t$ converges to the function $m(t) := \log(1 + t)/\log(2)$. This property is usually referred to as the Gauss-Kuzmin theorem. At the same time, Lévy in [9] proved the same theorem using a completely different method. The probability density $p(t) := 1/(\log(2)(1 + t))$ whose distribution function is $m(t)$ is usually known as Gauss' measure.

This theorem gives access to the frequency f_m of the digit m upon integrating the Gauss' measure $p(x)$ over the interval $[1/(m + 1); 1/m]$,

$$f_m = \frac{1}{\log 2} \log(1 + \frac{1}{m(m+2)}).$$

Finally, Khintchine in [6] proved that for almost all real x, the geometric mean of the quotients in the continued fraction expansion of x tends to a constant whose expression is

$$K_{SCF} := \lim_{n\to\infty} \sqrt[n]{q_1 q_2 \ldots q_n} = \prod_{m=1}^{\infty} \left(1 + \frac{1}{m(m+2)}\right)^{\frac{\log m}{\log 2}} \approx 2.685452001 \ldots.$$

This infinite product converges very slowly. Lehmer [8], Shanks and Wrench [14, 15], Gosper, Bailey, Borwein, and Crandall [1] provide several representations that make possible a precise numerical evaluation of K_{SCF}. The record currently belongs to Gourdon [5] who has determined the constant to 110,000 decimal places.

2.2 Centred continued fraction expansions

The principle of the centred Euclidean algorithm is to consider a pseudo-Euclidean division that involves the nearest integer rounding function

$$\lfloor x \rceil := \lfloor x + \frac{1}{2} \rfloor.$$

This corresponds to a continued fraction expansion for a real $-1/2 < x \leq 1/2$ of the form

$$x = \cfrac{\varepsilon_1}{q_1 + \cfrac{\varepsilon_2}{q_2 + \cfrac{\varepsilon_3}{q_3 + \cdots}}} = [\varepsilon_1 q_1, \varepsilon_2 q_2, \varepsilon_3 q_3, \ldots], \tag{1}$$

where $\varepsilon_i = \pm 1$, and $q_1, q_2, q_3, \ldots$ are strictly positive integers.

Precisely, the sequence $(\varepsilon_1 q_1, \varepsilon_2 q_2, \varepsilon_3 q_3, \ldots)$ of the partial quotients in the expansion is obtained by a combination of iterations of the shift function $T: \; (]-1/2, 1/2] \setminus \{0\}) \to (]-1/2, 1/2] \setminus \{0\})$ and the map function $\sigma: \; (]-1/2, 1/2] \setminus \{0\}) \to \mathbb{N}$, defined as follows

$$T(x) = \left|\frac{1}{x}\right| - \left\lfloor \left|\frac{1}{x}\right| \right\rceil, \qquad \text{and } \sigma(x) = \left\lfloor \left|\frac{1}{x}\right| \right\rceil.$$

This, together with the sign function $\operatorname{sgn}(x)$ provide the sequences $(q_1, q_2, q_3, \ldots)$ and $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots)$ associated to the real x in (1)

$$(q_1, q_2, q_3, \ldots) = (\sigma(x), \sigma(T(x)), \sigma(T^2(x)), \ldots), \text{ and}$$

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots) = (\operatorname{sgn}(x), \operatorname{sgn}(T(x)), \operatorname{sgn}(T^2(x)), \ldots).$$

In [10], Rieger proves a Gauss-Kuzmin theorem for the centred continued fraction expansion. The expansion (1) admits an invariant density of the Gauss' measure type. This measure can be found in Rieger [10].

PROPERTY 1. (Rieger) The invariant measure of the centred continued fraction expansion has density

$$p(x) = \begin{cases} \dfrac{1}{\log \phi} \dfrac{1}{\phi^2 + x} & \text{if } -1/2 \leq x < 0, \\ \dfrac{1}{\log \phi} \dfrac{1}{\phi + x} & \text{if } 0 \leq x \leq 1/2, \end{cases} \qquad \text{where } \phi := \frac{1 + \sqrt{5}}{2}.$$

By integrating this density for both the positive and the negative case, one obtains the frequency f_m of the digit m.

PROPERTY 2. (Rockett) The frequency f_m of digit m equals

$$f_m = \begin{cases} \dfrac{1}{\log\phi}\log\dfrac{3+5\phi}{2+5\phi}, & \text{if } m=2, \\ \dfrac{1}{\log\phi}\log\left(\dfrac{\phi(m-\frac{1}{2})+1}{\phi(m+\frac{1}{2})+1}\dfrac{\phi^2(m+\frac{1}{2})-1}{\phi^2(m-\frac{1}{2})-1}\right), & \text{otherwise.} \end{cases}$$

This expression for the frequency of digit m provides a representation of the Khintchine constant for the centred continued fraction expansion. This Khintchine constant is defined as the alsmost sure limit of the geometric mean of the absolute values of partial quotients in the centred continued fraction expansion.

COROLLARY. (Rockett) The centred Khintchine constant admits the following expression

$$K_{CCF} = \left(\frac{3+5\phi}{2+5\phi}\right)^{\frac{\log 2}{\log\phi}} \prod_{m\geq 3}\left(\frac{\phi(m-\frac{1}{2})+1}{\phi(m+\frac{1}{2})+1}\frac{\phi^2(m+\frac{1}{2})-1}{\phi^2(m-\frac{1}{2})-1}\right)^{\frac{\log m}{\log\phi}}. \tag{2}$$

This expression as well as the expression for the frequency of the digits have been given by Rockett in [11].

The infinite product converges very slowly. We give in the sequel several alternative expressions of this constant in order to obtain a precise numerical evaluation of K_{CCF}.

3 Evaluation of K_{CCF}

3.1 Expression of K_{CCF} involving the ζ' function

First, remark that the frequency f_m is the value at $1/m$ of a complex function $\psi(z) := \sum_{n\geq 2} a_n z^n$, that is analytic at 0. This leads to an expression of K_{CCF} by means of the derivative of the Riemann zeta function,

$$\log\phi\log K_{CCF} = \log 2\log\left(\frac{3+5\phi}{2+5\phi}\right) - \log\phi\sum_{n\geq 2} a_n(\zeta'(n) - \frac{\log 2}{2^n}),$$

where $\zeta'(n) := -\sum_{m\geq 1}\log(m)/m^n$. It proves convenient to introduce an integer parameter N in order to decrease the number of ζ' evaluations.

PROPOSITION 1. Let N be an arbitrary positive integer. The centred Khintchine constant is expressible in terms of the ζ' tail function

$$\zeta'(n,N) := \sum_{i=N+1}^{\infty}\frac{\log i}{i^n} = \zeta'(n) - \sum_{i=2}^{N}\frac{\log i}{i^n}$$

under the form

$$\log\phi\log K_{CCF} = \log 2\log\left(\frac{3+5\phi}{2+5\phi}\right) + \sum_{m=3}^{N} f_m\log(m) - \sum_{n\geq 2} a_n\zeta'(n,N), \tag{3}$$

where $\psi(1/m) := \sum_{n\geq 2} a_n/m^n$ is the expansion of f_m.

A fast approximation of the first 10 digits of K_{CCF} is obtained by taking $N = 2000$, $a_2 = 2.078086920$ and $a_3 = -0.4905697760$. Due to lack of sufficiently fast algorithms dedicated to computing the values of the ζ' function, its proves useful to deal with the Riemann zeta function instead of its derivative.

3.2 Expression of K_{CCF} involving the ζ function

We give here an expression of K_{CCF} by means of the Riemann ζ function for which fast evaluation algorithms are known (see Borwein [2]).

THEOREM 1. The centred Khintchine constant is expressible in terms of the ζ function

$$\zeta(n) := \sum_{i=1}^{\infty} \frac{1}{i^n}$$

under the form

$$\begin{aligned} & \log\phi \log K_{CCF} \\ = \quad & \log 3 \log\phi + \log\frac{2}{3}\log\left(\frac{5\phi+3}{5\phi+2}\right) \\ & + \sum_{n=2}^{\infty} \frac{(-1)^n}{n}(\zeta(n) - 1 - \frac{1}{2^n})\left[\lambda_1^n h_n(\frac{1}{\lambda_1}) + h_n(\lambda_1) - \lambda_2^n h_n(\frac{1}{\lambda_2}) - h_n(\lambda_2)\right] \quad (4) \end{aligned}$$

where $\lambda_1 := (\phi+2)/(2\phi)$ and $\lambda_2 := 1/(2\phi^3)$ involve the golden ratio $\phi := (\sqrt{5}+1)/2$ and h_n is the harmonic function

$$h_n(x) := \sum_{k=1}^{n-1} \frac{x^k}{k}.$$

PROOF. First, Abel's summation formula

$$A_N = S_{N+1} b_N - \sum_{k=3}^{N} S_n(b_{n+1} - b_n),$$

with

$$S_n := \sum_{k=3}^{n} \log\left(\frac{(k-\frac{1}{2}) + \frac{1}{\phi}}{(k+\frac{1}{2}) + \frac{1}{\phi}} \frac{(k+\frac{1}{2}) - \frac{1}{\phi^2}}{(k-\frac{1}{2}) - \frac{1}{\phi^2}}\right), \qquad b_n := \log n,$$

applies to the partial sum A_N of the second term in the expression (2) of K_{CCF}. The sum S_n simplifies to

$$S_n = \log\left(\phi\frac{5\phi+3}{2\phi+2} \cdot \frac{(n+\frac{3}{2}) - \frac{1}{\phi^2}}{(n+\frac{3}{2}) + \frac{1}{\phi}}\right) = \log\left(\phi\frac{5\phi+3}{2\phi+2} \cdot \frac{1 + (\frac{1}{2\phi^3})\frac{1}{n+1}}{1 + (\frac{\phi+2}{2\phi})\frac{1}{n+1}}\right).$$

Thus,

$$\begin{aligned} A_N &= \log\left(\phi\frac{5\phi+3}{2\phi+2}\right)\left[\log N - \sum_{k=3}^{N}\log(1+\frac{1}{k})\right] + \log\left(\frac{(N+\frac{3}{2})-\frac{1}{\phi^2}}{(N+\frac{3}{2})+\frac{1}{\phi}}\right)\log N \\ &+ \sum_{k=3}^{N}\log(1+\frac{1}{k})\left[\log(1+(\frac{1}{2\phi^3})\frac{1}{k}) - \log(1+(\frac{\phi+2}{2\phi})\frac{1}{k})\right]. \end{aligned} \tag{5}$$

Then, by taking the limit when N tends to ∞, one has

$$\lim_{N\to\infty}(\log N - \sum_{k=3}^{N}\log(1+\frac{1}{k})) = \log 3, \text{ and } \lim_{N\to\infty}\log N \log\left(\frac{(N+\frac{3}{2})-\frac{1}{\phi^2}}{(N+\frac{3}{2})+\frac{1}{\phi}}\right) = 0.$$

Finally, the last term of the summation (5) involves two terms of the form $\log(1+x)\log(1+\lambda_i x)$ with $\lambda_1 = (\phi+2)/(2\phi)$ and $\lambda_2 = 1/(2\phi^3)$. This term admits the expansion

$$\log(1+x)\log(1+\lambda x) = \sum_{n=2}^{\infty}\frac{(-1)^n}{n}[\lambda^n h_n(\frac{1}{\lambda}) + h_n(\lambda)]x^n, \text{ where } h_n(x) := \sum_{k=1}^{n-1}\frac{x^k}{k}.$$

This leads to formula (4) in the statement of Theorem 1.

Notice that the Leibniz theorem for alternating series applies. Thus, an approximation of the Khintchine constant, upon using the first n terms of the sum implies an error term of the form ρ^n with $\rho < 0.56$. Thus each new term adds about 1/3 of a digit.

An integer parameter N can be introduced in order to decrease the number of evaluations of the zeta function as was indicated earlier for the formula (3) in the context of the ζ' function. We have

$$\begin{aligned} &\log\phi\log K_{CCF} \\ =\ &\log 3\log\phi + \log\frac{2}{3}\log\left(\frac{5\phi+3}{5\phi+2}\right) \\ &+\sum_{k=3}^{N}\log(1+\frac{1}{k})[\log(1+\frac{\lambda_1}{k}) - \log(1+\frac{\lambda_2}{k})] \\ &+\sum_{n=2}^{\infty}\frac{(-1)^n}{n}\zeta(n,N+1)\left[\lambda_1^n h_n(\frac{1}{\lambda_1}) + h_n(\lambda_1) - \lambda_2^n h_n(\frac{1}{\lambda_2}) - h_n(\lambda_2)\right], \end{aligned} \tag{6}$$

where

$$\lambda_1 := \frac{\phi+2}{2\phi}, \qquad \lambda_2 := \frac{1}{2\phi^3}, \qquad \phi := \frac{\sqrt{5}+1}{2},$$

$$h_n(x) := \sum_{k=1}^{n-1}\frac{x^k}{k}.$$

There $\zeta(n, N)$ is the standard Hurwitz function

$$\zeta(n, N) = \sum_{i=N}^{\infty} \frac{1}{i^n}.$$

This trick has been previously used by Flajolet and Vardi [4] in the context of the standard Khintchine constant.

3.3 Numerical Evaluation

The expression (6) of K_{CCF} allows a fast computation of the centred Khintchine constant to 1000 digits. Take $N = 20$ and 900 terms of the m in (6) and get:

```
5.454517244545585756966057724994381016973272416251347045398035204159
8481492245344570465518924282365208908604640323788499860315783122561O
0646599715467892433625687187014720059591816277216755653672157920603l
8137584000715940199473403186326073700578837334101104696468912170929 6
1080855642533849185627002326768243615809078241454228858477377338845 2
6375510741623845008337865456878210510914449135355504587850469455761 5
1526024529907215944083910506539103023453734297572686592339909964587 9
4675559587716999010968167906220552278367119403594032057195600507482 5
3459834247391839985545090776111281263060442585297915949661023638527 0
0989385673791927720475422791641994398337283475772784382908656263135 4
2275976109065020523820384409430720267454249413386781230786344700686 6
6430106185537058130749597696000637242752799178902053811502778680118 6
1431679704207353087869905063318700953406926954181327511763584598915 9
9730542078562444412350236523944398662144465519119652014709794945351 8
840349914318260839373942055326804758017201997962 0
```

The computation needs about $1 \cdot 10^{11}$ elementary operations (3 minutes on a 500 Mhz machine in 2001).

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